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*-Solutions of Evolution Equations in Hilbert Space

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1. INTRODUCTION

In this section we will explain the objective of this paper and the motivation behind it.

Let H be a real Hilbert space with inner product and norm denoted by $(\cdot, \cdot)$ and $\|\cdot\|$. We consider on H the semilinear evolution equation

$$\begin{aligned} du(t)/dt &= A(t)\,u(t) + g(u, t), \qquad 0 \leqslant s < t \leqslant T, \\ u(s) &= x, \end{aligned} \tag{1}$$

where $\{A(t)\colon 0 \leqslant t \leqslant T\}$ is a family of closed linear operators on H whose domains $\mathscr{D}(A(t))$ are dense for each t, $g(t) \equiv g(u, t)$ may depend on $\{u(r)\colon 0 \leqslant r \leqslant T\}$ for each t, and $g \in L^2([0, T], H)$. Equation (1) is linear when g is independent of u. We refer to Pazy [9] and Tanabe [10] for linear evolution equations with the operators $A(t)$ assumed to be infinitesimal generators of C_0-semigroups.

A strong solution u of (1) requires $u(t) \in \mathscr{D}(A(t))$ for each t. It is possible that there are no strong solutions, but solutions of weaker types exist which do not need the abovementioned requirement (see [9, 10] in the

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linear case). For example, the homogeneous equation, that is with $g \equiv 0$, may have a strong solution but the inhomogeneous equation may not have a strong solution if $g(\cdot, t)$ does not belong to $\mathscr{D}(A(t))$. Let us recall the definitions of two classes of weaker solutions.

Suppose that $\{A(t): 0 \leqslant t \leqslant T\}$ generates a unique evolution operator $\{U(t, s): 0 \leqslant s \leqslant t \leqslant T\}$, i.e., the $U(t, s)$ are bounded linear operators on H such that

$$U(t, t) = I, \; U(t, s)\, U(s, r) = U(t, r) \qquad \text{for} \quad 0 \leqslant r \leqslant s \leqslant t \leqslant T$$

and

$$(t, s) \to U(t, s) \qquad \text{is strongly continuous for } 0 \leqslant s \leqslant t \leqslant T,$$

and certain relationships between $\{A(t)\}$ and $\{U(t, s)\}$ hold, which we will introduce later on (see Remark 2). Then u is called an *evolution solution* or *variation of constants solution* of (1) if it satisfies the evolution integral equation corresponding to (1), namely,

$$u(t) = U(t, s)\, x + \int_s^t U(t, r)\, g(u, r)\, dr, \qquad s \leqslant t \leqslant T.$$

In the linear case this solution exists and is unique.

Suppose the domains of the adjoint operators $A^*(t)$ are such that $D = \bigcap_{0 \leqslant t \leqslant T} \mathscr{D}(A^*(t))$ is dense in H. Then u is called a **-solution* of (1) if it is a solution of the *-equation corresponding to (1), namely,

$$(u(t), y) = (x, y) + \int_s^t (u(r), A^*(r)\, y)\, dr + \int_s^t (g(u, r), y)\, dr, \qquad s \leqslant t \leqslant T,$$

for all $y \in D$. (Solutions of this type are sometimes also called weak, but this term has several other meanings.)

The main question regarding evolution solutions and *-solutions, when they both exist, is when are they equivalent. If the adjoints $\{A^*(t)\}$ and $\{U^*(t, s)\}$ satisfy the relationships

$$\int_s^t (x, U^*(r, s)\, A^*(r)\, y)\, dr = (x, U^*(t, s)\, y) - (x, y) \tag{2}$$

for all $x \in H$ and $y \in D$, and, assuming $U^*(t, s)\, \mathscr{D}(A^*(t)) \subset \mathscr{D}(A^*(s))$,

$$\int_s^t (x, A^*(r)\, U^*(t, r)\, y)\, dr = (x, U^*(t, s)\, y) - (x, y) \tag{3}$$

for all $x \in H$ and $y \in \mathscr{D}(A^*(t))$, and certain other conditions hold, then it can be shown that evolution solutions and *-solutions are equivalent (Corollary 1). Observe, however, that the definition of evolution solution has nothing to do with the domains $\mathscr{D}(A^*(t))$, whereas the definition of *-solution and Eqs. (2) and (3) involve strong conditions on these domains.

Our objective in this paper is to define an extended *-solution of (1) which does not impose any restrictions on the $\mathscr{D}(A^*(t))$, and to prove that under appropriate extensions of Eqs. (2) and (3), which do not restrict the $\mathscr{D}(A^*(t))$ either, evolution solutions and extended *-solutions are also equivalent. Moreover, under the restrictions on the $\mathscr{D}(A^*(t))$ in the previous paragraph the extended definitions and assumptions coincide with the restricted ones. We will give a simple example of an evolution equation which is covered by the extended case but for which $U^*(t,s)\ \mathscr{D}(A^*(t)) \subset \mathscr{D}(A^*(s))$ does not holds and therefore is not covered by the restricted one.

Clearly in the linear case if an extended (or a restricted) *-solution exists and the equivalence holds, then it is unique.

Our interest in evolution solutions and *-solutions is motivated by the theory of stochastic evolution equations. Let us consider the stochastic evolution equation

$$\begin{aligned} du(t) &= A(t)\, u(t)\, dt + dM(u, t), \qquad 0 \leqslant s < t \leqslant T, \\ u(s) &= x, \end{aligned}$$

where $\{A(t)\}$ is as above and M is a square-integrable H-valued martingale such that $M(t) \equiv M(u, t)$ may depend on $\{u(r): 0 \leqslant r \leqslant t\}$ (only up to time t since M must be adapted). The deterministic theory is not applicable because M is not differentiable in the usual sense. Nevertheless one may consider strong solutions (in integrated form, with $u(t) \in \mathscr{D}(A(t))$ for almost all t), evolution solutions,

$$u(t) = U(t, s)\, x + \int_s^t U(t, r)\, dM(u, r),$$

and *-solutions satisfying the *-equation

$$(u(t), y) = (x, y) + \int_s^t (u(r), A^*(r)\, y)\, dr + (M(u, t) - M(u, s), y)$$

with $y \in D$, if D is dense, or extended *-solutions. The integral in the evolution solution is a stochastic integral; stochastic integrals of this type as functions of t have been studied by Kotelenez [7] and others. Evolution solutions are useful in the linear case because they can be used to investigate structural properties of solutions (e.g., Zabczyk [11]). On the other hand, *-solutions (when D is dense) have the property that the right-hand side of the *-equation is a semimartingale; it is "almost" a semimartingale in the extended case. The relevance of this fact is that the semimartingale property is very useful in stochastic analysis (see, e.g., Metivier [8]).

In the semigroup case, i.e., when $A = A(t)$ is independent of t and generates a C_0-semigroup, the equivalence of evolution solutions and *-solutions of stochastic evolution equations was proved by Chojnowska-Michalik [4]. The equivalence of solutions in the deterministic, linear, semigroup case is contained in a result of Ball [2] (see also Ball [3] for the semilinear, semigroup case). The semigroup case affords the advantage that A commutes with the semigroup. Related results concerning the relationship between solutions to (1) in the sense of distributions and evolution solutions have been obtained by Arosio [1]. We will treat the general evolution case for stochastic evolution equations elsewhere.

In Section 2 we give the definitions and results, and in Section 3 some examples. Section 4 contains the proofs.

2. Definitions and Results

We consider first a restricted case. Let us suppose that $D_1 = \bigcap_{0 \leqslant t \leqslant T} \mathscr{D}(A(t))$ and $D_2 = \bigcap_{0 \leqslant t \leqslant T} \mathscr{D}(A^*(t))$ are both dense in H. We will assume that

$$A(\cdot)\, y \in L^2([0, T], H) \qquad \text{for all } y \in D_1$$

and

$$A^*(\cdot)\, y \in L^2([0, T], H) \qquad \text{for all } y \in D_2.$$

The following terminology concerning relationships between $\{A(t)\}$ and $\{U(t, s)\}$, and between their adjoints, is useful.

Definition 1. UF: $\{U(t, s)\}$ is a *uniform forward evolution operator* if $U(t, s)\, H \subset \mathscr{D}(A(t))$ for $0 \leqslant s < t \leqslant T$ and

$$\int_s^t A(r)\, U(r, s)\, dr = U(t, s) - I \qquad \text{for all } s < t.$$

(Bochner integral in $\mathscr{L}(H)$, the bounded linear operators on H with operator norm $\|\cdot\|_{\mathscr{L}}$.)

SF: $\{U(t, s)\}$ is a *strong forward evolution operator if* $U(t, s)\ \mathscr{D}(A(s)) \subset \mathscr{D}(A(t))$ for $0 \leqslant s < t \leqslant T$ and

$$\int_s^t A(r)\ U(r, s)\ y dr = U(t, s)\ y - y \qquad \text{for all } s < t \text{ and } y \in \mathscr{D}(A(s))$$

(Bochner integral in H).

WF: $\{U(t, s)\}$ is a *weak forward evolution operator* if $U(t, s)\ \mathscr{D}(A(s)) \subset \mathscr{D}(A(t))$ for $0 \leqslant s < t \leqslant T$ and

$$\int_s^t (x, A(r)\ U(r, s)\ y)\ dr = (x, U(t, s)\ y) - (x, y)$$

$$\text{for all } s < t,\ x \in H \text{ and } y \in \mathscr{D}(A(s)).$$

Clearly UF $\Rightarrow$ SF $\Rightarrow$ WF, and WF plus $A(\cdot)\ U(\cdot, s)\ y \in L^1([0, T], H)$ for $y \in \mathscr{D}(A(s)) \Rightarrow SF$.

DEFINITION 2. UB: $\{U(t, s)\}$ is a *uniform backward evolution operator* if

$$\int_s^t U(t, r)\ A(r)\ dr = U(t, s) - I \qquad \text{for all } s < t$$

(Bochner integral in $\mathscr{L}(H)$, and $U(t, r)\ A(r)$ is interpreted as an extension to $\mathscr{L}(H)$, see [10]).

SB: $\{U(t, s)\}$ is a *strong backward evolution operator* if

$$\int_s^t U(t, r)\ A(r)\ y dr = U(t, s)\ y - y \qquad \text{for all } s < t \text{ and } y \in D_1$$

(Bochner integral in H).

WB: $\{U(t, s)\}$ is a *weak backward evolution operator* if

$$\int_s^t (x, u(t, r)\ A(r)\ y)\ dr = (x, U(t, s)\ y) - (x, y) \text{ for all } s < t,\ x \in H \text{ and } y \in D_1.$$

We have UB $\Rightarrow$ SB $\Rightarrow$ WB, and WB $\Rightarrow$ SB because $\sup_{0 \leqslant s \leqslant t \leqslant T} \|U(t, s)\|_{\mathscr{L}} < \infty$ (implied by the strong continuity of $(t, s) \to U(t, s)$) and $A(\cdot)\ y \in L^2([0, T], H) \Rightarrow U(t, \cdot)\ A(\cdot)\ y \in L^1([0, T], H)$.

DEFINITION 3. UFA: $\{U^*(t, s)\}$ is a *uniform forward adjoint evolution operator* if

$$\int_s^t U^*(r, s)\ A^*(r)\ dr = U^*(t, s) - I \qquad \text{for all } s < t.$$

($U^*(r, s)\ A^*(r)$ is interpreted as an extension to $\mathscr{L}(H)$.)

SFA: $\{U^*(t,s)\}$ is a *strong forward adjoint evolution operator* if

$$\int_s^t U^*(r,s)\,A^*(r)\,y\,dr = U^*(t,s)\,y - y \qquad \text{for all } s<t \text{ and } y\in D_2.$$

WFA: $\{U^*(t,s)\}$ is a *weak forward adjoint evolution operator* if

$$\int_s^t (x, U^*(r,s)\,A^*(r)\,y)\,dr = (x, U^*(t,s)\,y) - (x,y)$$

$$\text{for all } s<t,\ x\in H \text{ and } y\in D_2.$$

Note that UFA $\Rightarrow$ SFA $\Rightarrow$ WFA, and WFA $\Rightarrow$ SFA since

$$\sup_{0\leqslant s\leqslant t\leqslant T} \|U^*(t,s)\|_{\mathscr{L}} < \infty \qquad \text{and} \qquad A^*(\cdot)\,y\in L^2([0,T],H)$$

$$\Rightarrow U^*(\cdot,s)\,A^*(\cdot)\,y\in L^1([0,T],H).$$

DEFINITION 4. UBA: $\{U^*(t,s)\}$ is a *uniform backward adjoint evolution operator* if $U^*(t,s)\,H\subset D(A^*(s))$ for $0\leqslant s<t\leqslant T$ and

$$\int_s^t A^*(r)\,U^*(t,r)\,dr = U^*(t,s) - I \qquad \text{for all } s<t.$$

SBA: $\{U^*(t,s)\}$ is a *strong backward adjoint evolution operator* if $U^*(t,s)\,\mathscr{D}(A^*(t))\subset\mathscr{D}(A^*(s))$ for $0\leqslant s<t\leqslant T$ and

$$\int_s^t A^*(r)\,U^*(t,r)\,y\,dr = U^*(t,s)\,y - y \qquad \text{for all } s<t \text{ and } y\in\mathscr{D}(A^*(t)).$$

WBA: $\{U^*(t,s)\}$ is a *weak backward adjoint evolution operator* if $U^*(t,s)\,\mathscr{D}(A^*(t))\subset\mathscr{D}(A^*(s))$ for $0\leqslant s<t\leqslant T$ and

$$\int_s^t (x, A^*(r)\,U^*(t,r)\,y)\,dr = (x, U^*(t,s)\,y) - (x,y)$$

$$\text{for all } s<t,\ x\in H \text{ and } y\in\mathscr{D}(A^*(t)).$$

We have UBA $\Rightarrow$ SBA $\Rightarrow$ WBA, and WBA plus $A^*(\cdot)\,U^*(t,\cdot)\ y\in L^1([0,T],H)\Rightarrow$ SBA.

Remarks: (1) These definitions can be given more generally, without assuming D_1 and D_2 are dense, but for our purposes we need the denseness.

(2) The statement made in the Introduction that $\{A(t)\}$ generates a unique evolution operator $\{U(t,s)\}$ should be interpreted to mean that

$\{U(t, s)\}$ is the unique evolution operator which satisfies whichever of the defining properties above we need.

(3) Sufficient conditions are known on the $A(t)$, or on bilinear forms defining them, which guarantee the existence of a unique evolution operator $\{U(t, s)\}$ satisfying certain of the definitions above (see [9, 10]).

(4) UF $\Rightarrow$ WFA, UBA $\Rightarrow$ WB, UB $\Rightarrow$ WBA if $U^*(t, s)\ \mathscr{D}(A^*(t)) \subset \mathscr{D}(A^*(s))$ (because D_1 is dense), and UFA $\Rightarrow$ WF if $U(t, s)\ \mathscr{D}(A(s)) \subset \mathscr{D}(A(t))$ (because D_2 is dense).

Let us recall the definitions of solutions we need.

DEFINITION 5. $u \in C([0, T], H)$ is an *evolution solution* of (1) if it satisfies the *evolution integral equation*

$$u(t) = U(t, s)\, x + \int_s^t U(t, r)\, g(u, r)\, dr, \qquad s \leqslant t \leqslant T.$$

DEFINITION 6. $u \in C([0, T], H)$ is a *$*$-solution* of (1) if it satisfies the *$*$-equation*

$$(u(t), y) = (x, y) + \int_s^t (u(r), A^*(r)\, y)\, dr + \int_s^t (g(u, r), y)\, dr,$$
$$s \leqslant t \leqslant T \qquad \text{for all } y \in D_2.$$

We now give the relationships between evolution solutions and $*$-solutions.

PROPOSITION 1. *If u is an evolution solution and $\{U^*(t, s)\}$ is a* WFA, *then u is a $*$-solution.*

In order to obtain a converse result we need the following assumption.

ASSUMPTION 1. Let $\{U^*(t, s)\}$ be a WBA. We assume that there exists a dense subset D_3 of H such that $D_3 \subset D_2$, $A^*(s)\ U^*(t, s)\ D_3 \subset D_2$ for $0 \leqslant s < t \leqslant T$, and $\{U^*(t, s)\}$ satisfies

$$\int_s^t (x, A^*(s)\, A^*(r)\, U^*(t, r)\, y)\, dr = (x, A^*(s)\, U^*(t, s)\, y) - (x, A^*(s)\, y)$$

for all $s < t$, $x \in H$, and $y \in D_3$.

This assumption is satisfied if $\{U^*(t, s)\}$ is an SBA and $A^*(\cdot)\, U^*(t, \cdot)\, y$ is Bochner integrable with values in $\mathscr{D}(A^*(s))$ equipped with the corresponding graph norm, for $y \in D_3$.

PROPOSITION 2. *If u is a *-solution and $\{U^*(t,s)\}$ is a* WBA *satisfying Assumption* 1, *then u is an evolution solution.*

The following consequences are immediate.

COROLLARY 1. *If $\{U^*(t,s)\}$ is a* WFA *and a* WBA *satisfying Assumption* 1, *then evolution solutions and *-solutions are equivalent.*

COROLLARY 2. *In the linear case, under the assumptions of Corollary* 1 *if a *-solution exists it is unique.*

Remark 5. Since all the conditions of the previous results are stated in terms of adjoints, it is not necessary to require that D_1 be dense. However, if D_1 is dense, due to Remark 4 we can replace WBA by UB (if $U^*(t,s)\,\mathscr{D}(A^*(t)) \subset \mathscr{D}(A^*(s))$ as condition in Proposition 2. WFA can always be replaced by UF. On the other hand, the assumption on the $\mathscr{D}(A^*(t))$ are in general not implied by conditions on the $\mathscr{D}(A(t))$.

In the semigroup case, where $A = A(t)$ is independent of t and generates a C_0-semigroup $\{T(t)\}$, then $U(t,s) = T(t-s)$, $\{U(t,s)\}$ is a UF and a UB, $\{U^*(t,s)\}$ is a UFA and a UBA, and all the previous results hold with $D_1 = \mathscr{D}(A)$, $D_2 = \mathscr{D}(A^*)$, and $D_3 = \mathscr{D}((A^*)^2)$.

We now turn to the extended case, where no assumptions are made on the domains of $A(t)$ and $A^*(t)$ (in particular $\bigcap_{t\in[0,T]} \mathscr{D}(A^*(t))$ is not assumed to be dense). Clearly some of the previous conditions regarding the evolution operator $\{U(t,s)\}$ and the definition of *-solution do not make sense now. We take a different approach, which starts by defining a single operator $\mathscr{A}$ on $L^2([0,T],H)$ in terms of the family $\{A(t), 0 \leqslant t \leqslant T\}$.

DEFINITION 7. Given the family $\{A(t)\colon\ 0 \leqslant t \leqslant T\}$, an operator $\mathscr{A}\colon f \to [\mathscr{A}f](\cdot)$ from $L^2([0,T],H)$ into itself is defined with domain

$$\mathscr{D}(\mathscr{A}) = \{f \in L^2([0,T],H)\colon f(t) \in \mathscr{D}(A(t)) \text{ for } 0 \leqslant t \leqslant T,$$
$$\text{and } A(\cdot)\,f(\cdot) \in L^2([0,T],H)\},$$

and

$$[\mathscr{A}f](t) = A(t)\,f(t), \qquad 0 \leqslant t \leqslant T \text{ for } f \in \mathscr{D}(\mathscr{A}).$$

PROPOSITION 3. *$\mathscr{A}$ is closed.*

We will assume that $\mathscr{A}$ satisfies:

ASSUMPTION 2. $\mathscr{D}(\mathscr{A})$ is dense in $L^2([0,T],H)$.

This assumption is fairly general. For example, it is easy to see that it holds under the conditions of the restricted case above, i.e., D_1 is dense and $A(\cdot)\, y \in L^2([0, T], H)$ for $y \in D_1$.

Since $L^2([0, T], H)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_0^T (f(t), g(t))\, dt,$$

a standard consequence (e.g., [9, Lemma 10.5]) of Proposition 3 and Assumption 2 is

PROPOSITION 4. *The adjoint operator* $\mathscr{A}^*$: $f \to [\mathscr{A}^* f](\cdot)$ *from* $L^2([0, T], H)$ *into itself has dense domain* $\mathscr{D}(\mathscr{A}^*)$.

Remark 6. $\mathscr{A}^*$ is an extension of the family of operators $\{A^*(t)$: $0 \leqslant t \leqslant T\}$ in the sense that if $h \in L^2([0, T], H)$ is such that $h(t) \in \mathscr{D}(A^*(t))$ for all $t \in [0, T]$ and $A^*(\cdot)\ h(\cdot) \in L^2([0, T], H)$, then $h \in \mathscr{D}(\mathscr{A}^*)$ and $[\mathscr{A}^* h](t) = A^*(t)\, h(t)$ for $t \in [0, T]$.

Notation. Given $v \in H$ and $\varepsilon > 0$, by Proposition 4 there exists $h \in \mathscr{D}(\mathscr{A}^*)$ such that $\int_0^T \|h(t) - v\|^2\, dt < \varepsilon$, and we denote $h = h_v^\varepsilon$.

We will now define relationships between $\{U^*(t, s)\}$ and $\mathscr{A}^*$ corresponding to the definitions of WFA and WBA, and introduce an assumption analogous to Assumption 1.

DEFINITION 8. EWFA: $\{U^*(t, s)\}$ is an *extended weak forward adjoint evolution operator* if given $x, y \in H$ and $\varepsilon > 0$ there exists $h_y^\varepsilon \in \mathscr{D}(\mathscr{A}^*)$ such that

$$\int_s^t (x, U^*(r, s)[\mathscr{A}^* h_y^\varepsilon](r))\, dr = (x, U^*(t, s)\, y) - (x, y) + e_{x,y}^{1,\varepsilon}(s, t)$$

for all $s < t$, where $e_{x,y}^{1,\varepsilon}(s, t) \to 0$ as $\varepsilon \to 0$ uniformly for $s \in [0, t]$ and x in bounded sets in H. (Note that $e_{x,y}^{1,\varepsilon}(s, t)$ is measurable in x, s and t.)

DEFINITION 9. EWBA: $\{U^*(t, s)\}$ is an *extended weak backward adjoint evolution operator* if given $x, y \in H$ and $\varepsilon > 0$ there exists $h_{U^*(t,r)y}^\varepsilon \in \mathscr{D}(\mathscr{A}^*)$ for each $r \in [s, t)]$ satisfying $(r \to [\mathscr{A}^* h_{U^*(t,r)y}^\varepsilon](r)) \in L^2([0, t], H)$, and

$$\int_s^t (x, [\mathscr{A}^* h_{U^*(t,r)y}^\varepsilon](r))\, dr = (x, U^*(t, s)\, y) - (x, y) + e_{x,y}^{2,\varepsilon}(s, t)$$

for all $s < t$, where $e_{x,y}^{2,\varepsilon}(s, t) \to 0$ as $\varepsilon \to 0$ uniformly for $s \in [0, t]$ and x in bounded sets in H. (Note that $e_{x,y}^{2,\varepsilon}(s, t)$ is measurable in x and s.)

ASSUMPTION 3. Let $\{U^*(t, s)\}$ be an EWBA. We assume that there is a dense subset D of H such that given $x \in H$, $y \in D$, $\varepsilon > 0$ and $h_r \equiv h^{\varepsilon}_{U^*(t,r)y} \in \mathscr{D}(\mathscr{A}^*)$ satisfying Definition 9 for each $r \in [s, t]$, there exists $h^{\varepsilon}_{[\mathscr{A}^* h_r](r)} \in \mathscr{D}(\mathscr{A}^*)$ such that

$$((r, r') \to (x, [\mathscr{A}^* h^{\varepsilon}_{[\mathscr{A}^* h_r](r)}](r'))) \in L^1([s, t]^2, R)$$

and

$$\int_s^t (x, [\mathscr{A}^* h^{\varepsilon}_{[\mathscr{A}^* h_r](r)}](s))\, dr$$
$$= (x, [\mathscr{A}^* h^{\varepsilon}_{U^*(t,s)y}](s)) - (x, [\mathscr{A}^* h^{\varepsilon}_y](s)) + e^{3,\varepsilon}_{x,y}(s, t)$$

for all $s < t$, where $e^{3,\varepsilon}_{x,y}(s, t) \to 0$ as $\varepsilon \to 0$ and is uniformly bounded for $s \in [0, t]$ and x in bounded subsets of H. (Note that $e^{3,\varepsilon}_{x,y}(s, t)$ is measurable in x, s).

Remark 7. In the conditions for the restricted case all the measurabilities we need are automatic and the integrand in Assumption 1 is also jointly measurable in r, s. In the extended case the measurability of the integrand in Definition 8 is automatic but the measurabilities in Definition 9 and Assumption 3 must be assumed.

We now extend the definition of a *-solution.

DEFINITION 10. $u \in C([0, T], H)$ is an *extended *-solution* of (1) if for each $y \in H$ and $\varepsilon > 0$ there exists $h^{\varepsilon}_y \in \mathscr{D}(\mathscr{A}^*)$ such that u satisfies the *extended *-equation*

$$(u(t), y) = (x, y) + \int_s^t (u(r), [\mathscr{A}^* h^{\varepsilon}_y](r))\, dr + \int_s^t (g(u, r), y)\, dr + e^{\varepsilon}_{x,y}(s, t),$$

where $e^{\varepsilon}_{x,y}(s, t) \to 0$ as $\varepsilon \to 0$. (Note that $e^{\varepsilon}_{x,y}(s, t)$ is measurable in x, s, and t.)

It is easy to see that a strong solution is an extended *-solution and that in the restricted case above WFA and EWFA are equivalent, WBA and EWBA are equivalent, Assumptions 1 and 3 are equivalent, and *-solutions and extended *-solutions are equivalent.

Finally we give the relationships between evolution solutions and extended *-solutions and immediate corollaries.

DEFINITION 11. u is said to be a *bounded* (evolution or extended *) solution of (1) if $u(r)$ and $g(u, r)$ remain in bounded subsets of H for $r \in [0, T]$.

PROPOSITION 5. *If u is a bounded evolution solution and $\{U^*(t,s)\}$ is an* EWFA, *then u is an extended *-solution.*

PROPOSITION 6. *Let u be a bounded extended *-solution such that $e^{\varepsilon}_{x,y}(s,t)$ is measurable in y restricted to bounded subsets of H and $e^{\varepsilon}_{x,y}(s,t) \to 0$ as $\varepsilon \to 0$ uniformly for $t \in [s,T]$ and y in bounded subsets of H, and $\{U^*(t,s)\}$ be an* EBWA *satisfying Assumption* 3. *Assume in addition that for $y \in D$ and $\varepsilon > 0$, $[\mathscr{A}^* h^{\varepsilon}_{U^*(t,r)y}](r)$ remains bounded for $r \in [s,t]$. Then u is an evolution solution.*

COROLLARY 3. *Let $\{U^*(t,s)\}$ be an* EWFA *and an* EWBA. *Then under the conditions of Propositions* 5 *and* 6 *bounded evolution solutions and bounded *-solutions are equivalent.*

COROLLARY 4. *In the linear case, under the conditions of Corollary* 3, *if a bounded extended *-solution exists it is unique.*

Remark 8. We will show in Example 3.1 that Proposition 6 is false without Assumption 3.

Remark 9. All these results make sense if $A(t)$ is defined only for Lebesgue-almost all $t \in [0,T]$.

We will need the following result.

LEMMA 1. *Let $R(\lambda)$ denote the resolvent of the infinitesimal generator of a contraction semigroup on H. Then for each $x \in H$, $(x, \lambda R(\lambda)\, y - y) \to 0$ as $\lambda \to \infty$ uniformly for y in bounded subsets of H.*

3. EXAMPLES

EXAMPLE 3.1. Proposition 6 is false without Assumption 3.

Let $H = L^2[0,1]$, $\mathscr{D}(A_1^*) = \{f \in C^2[0,1]\colon f'(0) = f'(1) = 0\}$ and $\mathscr{D}(A_2^*) = \{f \in C^2[0,1]\colon f(0) = f(1) = 0\}$, with

$$A_1^* f(x) = \tfrac{1}{2} f''(x) \qquad \text{for} \quad f \in \mathscr{D}(A_1^*)$$

and

$$A_2^* f(x) = \tfrac{1}{2} f''(x) \qquad \text{for} \quad f \in \mathscr{D}(A_2^*).$$

A_1^* and A_2^* generate contraction semigroups on H which are distinct (reflecting and killing Brownian motion, respectively). We define the operators

$$A^*(t) = A_1^* 1_{[0, 1/2T]}(t) + A_2^* 1_{]1/2T, T]}(t), \qquad 0 \leqslant t \leqslant T,$$

and the adjoint evolution operator

$$U^*(t, s) = S^{(1)}_{t \wedge 1/2T - s \wedge 1/2T} S^{(2)}_{t \vee 1/2T - s \vee 1/2T}, \qquad 0 \leqslant s \leqslant t \leqslant T,$$

where $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, and $\{S_t^{(1)}\}$ and $\{S_t^{(2)}\}$ are the semigroups generated by A_1^* and A_2^*, respectively. It follows from Example 3.2 that $\{U^*(t, s)\}$ is an EWFA and an EWBA associated with $\{A^*(t)\}$.

The fact that $\{U^*(t, s)\}$ is an EWFA implies that $u(t) = U(t, s)\, x$ is an extended *-solution of the linear homogeneous equation $du(t)/dt = A(t)\, u(t)$, $u(s) = x$. Concerning the hypotheses of Proposition 6, we will see in Example 3.2 that $e^{\varepsilon}_{x,y}(s, t)$ is measurable in y restricted to bounded subsets of H, and converges to 0 as $\varepsilon \to 0$ uniformly for $t \in [s, T]$ and y in bounded sets; in addition, from the fact that $A^*(t)$ takes only two values it follows that $[\mathscr{A}^* h^{\varepsilon}_{U^*(t,r)y}](r)$ remains bounded for $r \in [s, t]$. However, it is clear that $u_1(t) = S^{(1)}_{t-s} x$ and $u_2(t) = S^{(2)}_{t-s} x$ are *-solutions ($\mathscr{D}(A_1^*) \cap \mathscr{D}(A_2^*)$ is dense), and hence extended *-solutions, of the same linear equation. Therefore we cannot conclude that the extended *-solution u is an evolution solution since this would guarantee uniqueness. We will see at the end of Example 3.2 that Assumption 3 fails in the present example.

EXAMPLE 3.2. EWBA and Assumption 3 do not imply WBA.

The purpose of this example is to illustrate that the extended results do cover cases not covered by the restricted ones, and we do this by constructing an EWBA that satisfies Assumption 3 but is not a WBA. (The other conditions of Proposition 6 are satisfied.)

Let H be a separable Hilbert space, and let $\{T_t^{(1)}\}$ and $\{T_t^{(2)}\}$ be two self-adjoint contraction semigroups on H whose corresponding infinitesimal generators A_1 and A_2 have compact resolvents $R_1(\lambda)$ and $R_2(\lambda)$, respectively. Then there exist orthonormal bases $\{x_n^1\} \subset \mathscr{D}(A_1)$ and $\{x_n^2\} \subset \mathscr{D}(A_2)$ of H, and reals $\lambda_n^{(j)}$, non-decreasing in n, $j = 1, 2$, such that $A_j x_n^j = -\lambda_n^{(j)} x_n^j$ and

$$R_j(\lambda)\, x_n^j = x_n^j/(\lambda + \lambda_n^{(j)}), \qquad n = 1, 2, ...; j = 1, 2, \text{ [6]}.$$

The semigroups can then be represented as

$$T_t^{(j)} x = \sum_{n=1}^{\infty} e^{-\lambda_n^{(j)} t} (x, x_n^j)\, x_n^j, \qquad j = 1, 2,$$

and

$$\mathscr{D}(A_j)=\left\{x\in H\colon \sum_{n=1}^{\infty} (\lambda_n^{(j)})^2(x, x_n^{(j)})^2<\infty\right\}, \qquad j=1, 2.$$

We define the operator

$$A(t)=A_1 1_{[0,1/2T]}(t)+A_2 1_{]1/2T,T]}(t), \qquad 0\leqslant t\leqslant T,$$

and the evolution operator

$$U(t, s)=T^{(2)}_{t\vee 1/2T-s\vee 1/2T}T^{(1)}_{t\wedge 1/2T-s\wedge 1/2T}, \qquad 0\leqslant s\leqslant t\leqslant T.$$

Clearly $A^*(t)=A(t)$ [9, Corollary 10.6], and the adjoint evolution operator is

$$U^*(t, s)=T^{(1)}_{t\wedge 1/2T-s\wedge 1/2T}T^{(2)}_{t\vee 1/2T-s\vee 1/2T}, \qquad 0\leqslant s\leqslant t\leqslant T.$$

We will verify that $\{U^*(t,s)\}$ is an EWFA and an EWBA associated with $\{A^*(t)\}$. We denote $t_0=\frac{1}{2}T$, and we will consider only the case $s\leqslant t_0<t$, which is of course the only nontrivial one.

EWFA: For $y\in H$ and $\varepsilon>0$, let

$$h_y^\varepsilon(t)=\lambda_\varepsilon^y R_1(\lambda_\varepsilon^y)\, y 1_{[0,t_0]}(t)+\lambda_\varepsilon^y R_2(\lambda_\varepsilon^y)\, y 1_{]t_0,T]}(t), \qquad 0\leqslant t\leqslant T,$$

where λ_ε^y is chosen so that

$$\|\lambda_\varepsilon^y R_j(\lambda_\varepsilon^y)\, y-y\|<(\varepsilon/T)^{1/2}, \qquad j=1, 2$$

[9, Lemma 3.2]. Then $h_y^\varepsilon\in\mathscr{D}(\mathscr{A}^*)$ $(=\mathscr{D}(\mathscr{A}))$, and for $x\in H$ we have

$$\begin{aligned}
&\int_s^t (x, U^*(r,s)[\mathscr{A}^*h_y^\varepsilon](r))\,dr\\
&\quad=\int_s^{t_0}(x, T^{(1)}_{r-s}A_1\lambda_\varepsilon^y R_1(\lambda_\varepsilon^y)\, y)\,dr+\int_{t_0}^t (x, T^{(1)}_{t_0-s}T^{(2)}_{r-t_0}A_2\lambda_\varepsilon^y R_2(\lambda_\varepsilon^y)\, y)\,dr\\
&\quad=(x, [T^{(1)}_{t_0-s}-I]\,\lambda_\varepsilon^y R_1(\lambda_\varepsilon^y)\, y)+(x, T^{(1)}_{t_0-s}[T^{(2)}_{t-t_0}-I]\,\lambda_\varepsilon^y R_2(\lambda_\varepsilon^y)\, y)\\
&\quad=(x, U^*(t,s)\, y)-(x, y)+e^{1,\varepsilon}_{x,y}(s,t),
\end{aligned}$$

where

$$\begin{aligned}
e^{1,\varepsilon}_{x,y}(s,t)&=(x, U^*(t,s)[\lambda_\varepsilon^y R_2(\lambda_\varepsilon^y)\, y-y])\\
&\quad+(x, T^{(1)}_{t_0-s}[\lambda_\varepsilon^y R_1(\lambda_\varepsilon^y)\, y-\lambda_\varepsilon^y R_2(\lambda_\varepsilon^y)\, y])\\
&\quad+(x, y-\lambda_\varepsilon^y R_1(\lambda_\varepsilon^y)\, y);
\end{aligned}$$

hence $e^{1,\varepsilon}_{x,y}(s,t)\to 0$ as $\varepsilon\to 0$ uniformly for $s\in[0,t]$ and x in bounded sets. Moreover, by Lemma 1 given $z\in H$, $K<\infty$ and $\delta>0$ there exists $\lambda_0>0$ such that $\lambda\geqslant\lambda_0$ implies $|(z,\lambda R_j(\lambda)\,y-y)|<\delta$ for all y such that $\|y\|\leqslant K$, $j=1,2$; hence a common value of λ^y_ε can be chosen for all y in a given bounded set, and it follows that $e^{1,\varepsilon}_{x,y}(s,t)$ is also measurable in y for y restricted to bounded sets, and converges to 0 as $\varepsilon\to 0$ uniformly for y in bounded sets.

Example 3.1 is a special case of Example 3.2. Hence $u(t)=U(t,s)\,x$ is an extended *-solution of the homogeneous equation with $e^{\varepsilon}_{x,y}(s,t)=e^{1,\varepsilon}_{x,y}(s,t)$ satisfying the condition of Proposition 6.

EWBA: For $y\in H$ and $\varepsilon>0$, let

$$h^{\varepsilon}_{U^*(t,r)y}(r')=y_\varepsilon(r,t)\,1_{[0,t_0]}(r')+T^{(2)}_{t-r}\lambda_\varepsilon R_2(\lambda_\varepsilon)\,y1_{]t_0,T]}(r')$$
$$\text{for}\quad r\in\,]t_0,T]$$

and

$$\begin{aligned}h^{\varepsilon}_{U^*(t,r)y}(r')&=T^{(1)}_{t_0-r}\lambda_\varepsilon R_1(\lambda_\varepsilon)\,T^{(2)}_{t-t_0}y1_{[0,t_0]}(r')\\&\quad+\lambda_\varepsilon R_2(\lambda_\varepsilon)\,T^{(1)}_{t_0-r}T^{(2)}_{t-t_0}y1_{]t_0,T]}(r')\qquad\text{for}\quad r\in[0,t_0],\end{aligned}$$

where $y_\varepsilon(r,t)\in\mathscr{D}(A_1)$ and λ_ε are chosen appropriately. We denote $h_r\equiv h^{\varepsilon}_{U^*(t,r)y}$ for short. Then $h_r\in\mathscr{D}(\mathscr{A}^*)$ and $(r\to[\mathscr{A}^*h_r](r))\in L^2([0,t],H)$. Now, for $x\in H$ we have

$$\begin{aligned}&\int_s^t(x,[\mathscr{A}^*h_r](r))\,dr\\&\quad=\int_s^{t_0}(x,A_1T^{(1)}_{t_0-r}\lambda_\varepsilon R_1(\lambda_\varepsilon)\,T^{(2)}_{t-t_0}y)\,dr+\int_{t_0}^t(x,A_2T^{(2)}_{t-r}\lambda_\varepsilon R_2(\lambda_\varepsilon)\,y)\,dr\\&\quad=(x,[T^{(1)}_{t_0-s}-I]\,\lambda_\varepsilon R_1(\lambda_\varepsilon)\,T^{(2)}_{t-t_0}y)+(x,[T^{(2)}_{t-t_0}-I]\,\lambda_\varepsilon R_2(\lambda_\varepsilon)\,y).\qquad\text{(a)}\end{aligned}$$

Hence

$$\int_s^t(x,[\mathscr{A}^*h_r](r))\,dr=(x,U^*(t,s)\,y)-(x,y)+e^{2,\varepsilon}_{x,y}(s,t),$$

where

$$\begin{aligned}e^{2,\varepsilon}_{x,y}(s,t)&=(x,T^{(1)}_{t_0-s}[\lambda_\varepsilon R_1(\lambda_\varepsilon)-I]\,T^{(2)}_{t-t_0}y)+(x,[I-\lambda_\varepsilon R_2(\lambda_\varepsilon)]\,y)\\&\quad+(x,[I-\lambda_\varepsilon R_1(\lambda_\varepsilon)]\,T^{(2)}_{t-t_0}y)+(x,T^{(2)}_{t-t_0}[\lambda_\varepsilon R_2(\lambda_\varepsilon)-I]\,y);\end{aligned}$$

hence $e^{2,\varepsilon}_{x,y}(s,t)\to 0$ as $\varepsilon\to 0$ uniformly for $s\in[0,t]$ and x in bounded sets.

Finally we consider Assumption 3. For h_r as above, let

$$h^{\varepsilon}_{[\mathscr{A}^*h_r](r)}(r') = \{\lambda'_\varepsilon R_1(\lambda'_\varepsilon)\, 1_{[0,t_0]}(r') + \lambda'_\varepsilon R_2(\lambda'_\varepsilon)\, 1_{]t_0,T]}(r')\}[\mathscr{A}^*h_r](r),$$
$$0 \leqslant r' \leqslant T,$$

with λ'_ε chosen appropriately; in fact we may choose $\lambda'_\varepsilon = \lambda_\varepsilon$. Then $h^{\varepsilon}_{[\mathscr{A}^*h_r](r)} \in \mathscr{D}(\mathscr{A}^*)$ and $((r, r') \to (x, [\mathscr{A}^*h^{\varepsilon}_{[\mathscr{A}^*h_r](r)}](r'))) \in L^1([s, t]^2, R)$.

Note that (a) implies

$$\int_s^t [\mathscr{A}^*h_r](r)\, dr = [T^{(1)}_{t_0-s} - I]\, \lambda_\varepsilon R_1(\lambda_\varepsilon)\, T^{(2)}_{t-t_0} y + [T^{(2)}_{t-t_0} - I]\, \lambda_\varepsilon R_2(\lambda_\varepsilon)\, y. \tag{b}$$

Now, since

$$\int_s^t (\|h^{\varepsilon}_{[\mathscr{A}^*h_r](r)}(s)\|^2 + \|A_1 h^{\varepsilon}_{[\mathscr{A}^*h_r](r)}(s)\|^2)^{1/2}\, dr < \infty,$$

then $\int_s^t h^{\varepsilon}_{[\mathscr{A}^*h_r](r)}(s)\, dr \in \mathscr{D}(A_1)$ and

$$A_1 \int_s^t h^{\varepsilon}_{[\mathscr{A}^*h_r](r)}(s)\, dr = \int_s^t A_1 h^{\varepsilon}_{[\mathscr{A}^*h_r](r)}(s)\, dr.$$

Hence, from (b) we have, for $x \in H$,

$$\begin{aligned}
&\int_s^t (x, [\mathscr{A}^*h^{\varepsilon}_{[\mathscr{A}^*h_r](r)}](s))\, dr \\
&\quad = \left(x, A_1 \int_s^t h_{[\mathscr{A}^*h_r](r)}(s)\, dr\right) \\
&\quad = \left(x, A_1 \lambda_\varepsilon R_1(\lambda_\varepsilon) \int_s^t [\mathscr{A}^*h_r](r)\, dr\right) \\
&\quad = (x, A_1 \lambda_\varepsilon R_1(\lambda_\varepsilon)\{[T^{(1)}_{t_0-s} - I]\, \lambda_\varepsilon R_1(\lambda_\varepsilon)\, T^{(2)}_{t-t_0} y \\
&\qquad + [T^{(2)}_{t-t_0} - I]\, \lambda_\varepsilon R_2(\lambda_\varepsilon)\, y\}) \\
&\quad = (x, [\mathscr{A}^*h^{\varepsilon}_{U^*(t,s)y}](s)) - (x, [\mathscr{A}^*h^{\varepsilon}_y](s)) + e^{3,\varepsilon}_{x,y}(s, t),
\end{aligned}$$

with $h^{\varepsilon}_y(r) = \lambda_\varepsilon R_1(\lambda_\varepsilon)\, y 1_{[0,t_0]}(r) + \lambda_\varepsilon R_2(\lambda_\varepsilon)\, y 1_{]t_0,T]}(r)$ and

$$\begin{aligned}
e^{3,\varepsilon}_{x,y}(s, t) &= (x, [\lambda_\varepsilon R_1(\lambda_\varepsilon) - I]\, A_1 \lambda_\varepsilon R_1(\lambda_\varepsilon)\, T^{(1)}_{t_0-s} T^{(2)}_{t-t_0} y) \\
&\quad - (x, [\lambda_\varepsilon R_1(\lambda_\varepsilon) - I]\, A_1 \lambda_\varepsilon R_1(\lambda_\varepsilon)\, T^{(2)}_{t-t_0} y) \\
&\quad + (x, A_1 \lambda_\varepsilon R_1(\lambda_\varepsilon)\, T^{(2)}_{t-t_0}[\lambda_\varepsilon R_2(\lambda_\varepsilon) - I]\, y) \\
&\quad - (x, A_1 \lambda_\varepsilon R_1(\lambda_\varepsilon)[\lambda_\varepsilon R_2(\lambda_\varepsilon) - I]\, y),
\end{aligned}$$

where we have used the commutativity of A_1 with $T_t^{(1)}$ and A_1 with $R_1(\lambda)$ on $\mathscr{D}(A_1)$, and also (in this case) $T_{t_0-s}^{(1)}$ with $R_1(\lambda)$.

In order to verify Assumption 3 we must see under what conditions $e_{x,y}^{3,\varepsilon}(s,t) \to 0$ as $\varepsilon \to 0$ and is uniformly bounded for $s \in [0, t]$ and x in bounded sets.

Let $y \in \mathscr{D}(A_2)$, $y_\lambda = [\lambda R_2(\lambda) - I]\, y$, and $y'_\lambda = T_{t-t_0}^{(2)}[\lambda R_2(\lambda) - I]\, y$. Then $A_1 \lambda R_1(\lambda)\, y_\lambda$ and $A_1 \lambda R_1(\lambda)\, y'_\lambda$ remain bounded as $\lambda \to \infty$. Indeed, $\|A_1 \lambda R_1(\lambda)\, y_\lambda\| \leqslant \|A_1 \lambda R_1(\lambda)\| \, \|y_\lambda\| \leqslant \lambda \|\lambda R_1(\lambda) - I\| \, \|A_2 y\|/\lambda \leqslant 2\|A_2 y\|$ [9, Lemma 3.2], and similarly for y'_λ. Hence to show that $(x, A_1 \lambda R_1(\lambda)\, y_\lambda)$ and $(x, A_1 \lambda R_1(\lambda)\, y'_\lambda)$ tend to 0 as $\lambda \to \infty$ for each $x \in H$ it suffices to verify that this holds for all x in a dense subspace of H [6, Lemma 1.31]; but this clearly holds for $x \in \mathscr{D}(A_1)$ $(= \mathscr{D}(A_1^*))$ since y_λ and y'_λ converge to 0 as $\lambda \to \infty$. Therefore the last two terms of $e_{x,y}^{3,\varepsilon}(s,t)$ tend to 0 as $\varepsilon \to 0$ for $y \in \mathscr{D}(A_2)$.

For the first two terms of $e_{x,y}^{3,\varepsilon}(s,t)$ we must show that they go to 0 as $\varepsilon \to 0$ for y in a dense set $D \subset \mathscr{D}(A_2)$. Since $[\lambda R_1(\lambda) - I]\; A_1 \lambda R_1(\lambda) = \lambda (A_1 R_1(\lambda))^2$, it suffices to prove that $\lambda (A_1 R_1(\lambda))^2\; z \to 0$ as $\lambda \to \infty$ for $z = T_{t-t_0}^{(2)} y$ and $z = T_{t_0-s}^{(1)} T_{t-t_0}^{(2)} y$ with $y \in D$. We will give an explicit example where this occurs.

Let $\lambda_n^{(1)} = e^n$ for all n. For each m, x_m^2 has representation of the form $x_m^2 = \sum_{n=1}^\infty b_{mn} x_n^1$. We assume $|b_{mn}| \leqslant K e^{-n}/n^{1/2}$ for all m and n and some constant K. Let D denote the set of all $y \in H$ that are finite linear combinations of the basis elements x_m^2; hence D is dense in H and $D \subset \mathscr{D}(A_2)$. Let $y = \sum_{m=1}^N a_m x_m^2 \in D$. Then both of the limits above hold if $\sum_{n=1}^\infty b_{mn} \lambda (A_1 R_1(\lambda))^2\; x_n^1 \to 0$ as $\lambda \to \infty$ for each m. But

$$\sum_{n=1}^{\infty} b_{mn} \lambda (A_1 R_1(\lambda))^2\, x_n^1 = \sum_{n=1}^{\infty} b_{mn} \lambda [\lambda_n^{(1)}/(\lambda + \lambda_n^{(1)})]^2\, x_n^1;$$

hence we must prove that $\lim_{\lambda \to \infty} \lambda^2 \sum_{n=1}^\infty b_{mn}^2 [\lambda_n^{(1)}/(\lambda + \lambda_n^{(1)})]^4 = 0$. We verify this as follows: for arbitrary N we have

$$\begin{aligned}
&\lim_{\lambda \to \infty} \lambda^2 \sum_{n=1}^{\infty} b_{mn}^2 [\lambda_n^{(1)}/(\lambda + \lambda_n^{(1)})]^4 \\
&\quad \leqslant K^2 \lim_{\lambda \to \infty} \lambda^2 \sum_{n=N}^{\infty} e^{2n}/(\lambda + e^n)^4\, n \\
&\quad \leqslant K' \lim_{\lambda \to \infty} \lambda^2 \int_N^\infty (e^{2x}/(\lambda + e^x)^4\, x)\, dx \\
&\quad \leqslant (K'/N) \lim_{\lambda \to \infty} \lambda^2 \int_N^\infty e^{2x}/(\lambda + e^x)^4\, dx \\
&\quad = (K'/N) \lim_{\lambda \to \infty} [\lambda^2/2(\lambda + e^N)^2 - \lambda^3/3(\lambda + e^N)^3] = K'/6N,
\end{aligned}$$

from which the desired result follows by letting $N \to \infty$.

Moreover, it is clear that $e^{3,\varepsilon}_{x,y}(s, t)$ is uniformly bounded for $s \in [0, t]$ and x in bounded sets, with $y \in D$. Therefore Assumption 3 is satisfied.

This example gives an EWBA which satisfies Assumption 3. We will now show that this EWBA is not necessarily a WBA. In order to have the WBA property, $U^*(t, t_0) = T^{(2)}_{t-t_0}$ with $t > t_0$ must map $\mathscr{D}(A_2)$ into $\mathscr{D}(A_1)$ (see Definition 4). This implies that $x^2_m \in \mathscr{D}(A_1)$ for all m, i.e.,

$$\sum_{n=1}^{\infty} (\lambda^{(1)}_n)^2 (x^2_m, x^1_n)^2 = \sum_{n=1}^{\infty} e^{2n} b^2_{mn} < \infty \qquad \text{for all } m;$$

but it is possible to choose b_{mn} so that this fails for any m.

Going back to Example 3.1, since it is a special case of Example 3.2, the corresponding adjoint evolution operation $\{U^*(t, s)\}$ is an EWBA. In order to show that it does not satisfy Assumption 3 it suffices to show that $e^{3,\varepsilon}_{x,y}(s, t)$ does not go to 0 as $\varepsilon \to 0$, and for this it suffices to prove that (first two terms of $e^{3,\varepsilon}_{x,y}(s, t)$)

$$[\lambda R_1(\lambda) - I]\, A_1 \lambda R_1(\lambda)\, x^2_m$$

becomes unbounded as $\lambda \to \infty$ for each x^2_m. In this case

$$x^1_n = \sqrt{2} \cos n\pi\theta, \qquad n = 1, 2, \ldots$$
$$x^2_m = \sqrt{2} \sin m\pi\theta, \qquad m = 1, 2, \ldots, 0 \leqslant \theta \leqslant 1.$$

Then $\lambda^{(1)}_n = n^2\pi^2/2$ and

$$[\lambda R_1(\lambda) - I]\, A_1 \lambda R_1(\lambda)\, x^2_m = \lambda (A_1 R_1(\lambda))^2 \sum_{n=1}^{\infty} (x^2_m, x^1_n)\, x^1_n$$
$$= \sum_{n=1}^{\infty} (x^2_m, x^1_n) [\lambda (\lambda^{(1)}_n)^2 / (\lambda + \lambda^{(1)}_n)^2]\, x^1_n;$$

hence

$$\| [\lambda R_1(\lambda) - I]\, A_1 \lambda R_1(\lambda)\, x^2_m \|^2 = \sum_{n=1}^{\infty} (x^2_m, x^1_n)^2 [\lambda (\lambda^{(1)}_n)^2 / (\lambda + \lambda^{(1)}_n)^2]^2.$$

Since $(x^2_m, x^1_n) \sim 2/n$ as $n \to \infty$ for fixed m, when m is even and n odd, or vice versa the series becomes unbounded as $\lambda \to \infty$.

4. Proofs

Proof of Proposition 3. Let $f_n \in \mathscr{D}(\mathscr{A})$, $f_n \to f$ and $\mathscr{A} f_n \to g$ in $L^2([0, T], H)$. Then there are subsequences (n_j) and (n_k) such that $f_{n_j}(t) \to f(t)$ and $[\mathscr{A} f_{n_k}](t) = A(t) f_{n_k}(t) \to g(t)$ for Lebesgue-almost all t in $[0, T]$. We may assume that $(n_j) = (n_k)$. Then for each t for which both limits hold $f(t) \in \mathscr{D}(A(t))$ and $A(t) f(t) = g(t)$ because $A(t)$ is closed. Hence $f \in \mathscr{D}(\mathscr{A})$ and $\mathscr{A} f = g$.

Proofs will only be given for Propositions 5 and 6 since the proofs ol Propositions 1 and 2 are almost identical.

Proof of Proposition 5. We denote $g(t) \equiv g(u, t)$. Since u is an evolution solution it satisfies:

$$(u(t), v) = (U(t, s)\, x, v) + \left(\int_s^t U(t, r)\, g(r)\, dr, v \right)$$

$$= (x, U^*(t, s)\, v) + \int_s^t (g(r), U^*(t, r)\, v)\, dr \tag{a}$$

for all $v \in H$. Let $y \in H$ and $\varepsilon > 0$. Fix $r \in [s, t]$. Put $v = [\mathscr{A}^* h_y^\varepsilon](r)$ and $t = r$ in (a) where h_y^ε is as in Definition 8, and integrate on r from s to t,

$$\int_s^t (u(r), [\mathscr{A}^* h_y^\varepsilon](r))\, dr$$

$$= \int_s^t (x, U^*(r, s)[\mathscr{A}^* h_y^\varepsilon](r))\, dr$$

$$+ \int_s^t \int_s^r (g(r'), U^*(r, r')[\mathscr{A}^* h_y^\varepsilon](r))\, dr'\, dr. \tag{b}$$

Applying Fubini's theorem in (b) we obtain

$$\int_s^t (u(r), [\mathscr{A}^* h_y^\varepsilon](r))\, dr$$

$$= \int_s^t (x, U^*(r, s)[\mathscr{A}^* h_y^\varepsilon](r))\, dr$$

$$+ \int_s^t \int_{r'}^t (g(r'), U^*(r, r')[\mathscr{A}^* h_y^\varepsilon](r))\, dr\, dr',$$

and by the EWFA property,

$$\int_s^t (u(r), [\mathscr{A}^* h_y^\varepsilon](r))\, dr$$

$$= (x, U^*(t, s)\, y) - (x, y) + e_{x,y}^{1,\varepsilon}(s, t)$$

$$+ \int_s^t (g(r'), U^*(t, r')\, y)\, dr' - \int_s^t (g(r'), y)\, dr'$$

$$+ \int_s^t e_{g(r'),y}^{1,\varepsilon}(r', t)\, dr'. \tag{c}$$

Substituting (a), with $v=y$, into (c), we obtain

$$\int_s^t (u(r), [\mathscr{A}^* h_y^\varepsilon](r))\, dr$$

$$= (u(t), y) - (x, y) - \int_s^t (g(r), y)\, dr + e^{1,\varepsilon}_{x,y}(s, t)$$

$$+ \int_s^t e^{1,\varepsilon}_{g(r'),y}(r', t)\, dr'$$

so that u is an extended $*$-solution with

$$e^\varepsilon_{x,y}(s, t) = -e^{1,\varepsilon}_{x,y}(s, t) - \int_s^t e^{1,\varepsilon}_{g(r),y}(r, t)\, dr,$$

since $g(r)$ remains in a bounded subset of H.

Proof of Proposition 6. Again we denote $g(t) \equiv g(u, t)$. Let $v \in H$ and $\varepsilon > 0$. Since u is an extended $*$-solution, it satisfies

$$(u(r), v) = (x, v) + \int_s^r (u(r'), [\mathscr{A}^* h_v^\varepsilon](r'))\, dr'$$

$$+ \int_s^r (g(r'), v)\, dr' + e^\varepsilon_{x,v}(s, r). \tag{a}$$

Let $y \in D$ and fix $r \in [s, t]$. Put $v = [\mathscr{A}^* h_r](r)$ into (a), where $h_r = h^\varepsilon_{U^*(t,r)y}$ as in Definition 9, integrate on r from s to t and apply Fubini's theorem

$$\int_s^t (u(r), [\mathscr{A}^* h_r](r))\, dr$$

$$= \int_s^t (x, [\mathscr{A}^* h_r](r))\, dr + \int_s^t \int_s^r (u(r'), [\mathscr{A}^* h^\varepsilon_{[\mathscr{A}^* h_r](r)}](r'))\, dr'\, dr$$

$$+ \int_s^t \int_s^r (g(r'), [\mathscr{A}^* h_r](r))\, dr'\, dr + \int_s^t e^\varepsilon_{x,[\mathscr{A}^* h_r](r)}(s, r)\, dr$$

$$= \int_s^t (x, [\mathscr{A}^* h_r](r))\, dr + \int_s^t \int_{r'}^t (u(r'), [\mathscr{A}^* h^\varepsilon_{[\mathscr{A}^* h_r](r)}](r'))\, dr\, dr'$$

$$+ \int_s^t \int_{r'}^t (g(r'), [\mathscr{A}^* h_r](r))\, dr\, dr' + \int_s^t e^\varepsilon_{x,[\mathscr{A}^* h_r](r)}(s, r)\, dr.$$

Then by the EWBA property and Assumption 3,

$$\int_s^t (u(r), [\mathscr{A}^*h_r](r))\,dr$$

$$= (x, U^*(t,s)\,y) - (x, y) + \int_s^t (u(r'), [\mathscr{A}^*h_{r'}](r'))\,dr'$$

$$- \int_s^t (u(r'), [\mathscr{A}^*h_y^\varepsilon](r'))\,dr' + \int_s^t (g(r'), U^*(t,r')\,y)\,dr'$$

$$- \int_s^t (g(r'), y)\,dr' + e_{x,y}^{2,\varepsilon}(s,t) + \int_s^t e_{u(r'),y}^{3,\varepsilon}(r',t)\,dr'$$

$$+ \int_s^t e_{g(r'),y}^{2,\varepsilon}(r',t)\,dr' + \int_s^t e_{x,[\mathscr{A}^*h_r](r)}^{\varepsilon}(s,r)\,dr. \tag{b}$$

Hence

$$\int_s^t (u(r), [\mathscr{A}^*h_y^\varepsilon](r))\,dr$$

$$= (x, U^*(t,s)\,y) - (x, y) + \int_s^t (g(r), U^*(t,r)\,y)\,dr$$

$$- \int_s^t (g(r), y)\,dr + e^\varepsilon, \tag{c}$$

where

$$e^3 = e_{x,y}^{2,\varepsilon}(s,t) + \int_s^t e_{u(r),y}^{3,\varepsilon}(r,t)\,dr + \int_s^t e_{g(r),y}^{2,\varepsilon}(r,t)\,dr$$

$$+ \int_s^t e_{x,[\mathscr{A}^*h_r](r)}^{\varepsilon}(s,r)\,dr.$$

Substituting (c) into (a), with $v = y$ and $r = t$, we obtain

$$(u(t), y) = (x, U^*(t,s)\,y) + \int_s^t (g(r), U^*(t,r)\,y)\,dr + e_{x,y}^{\varepsilon}(s,t) + e^\varepsilon. \tag{d}$$

Since (d) holds for all $\varepsilon > 0$ and $e^\varepsilon \to 0$ as $\varepsilon \to 0$, because $u(r)$, $g(r)$, and $[\mathscr{A}^*h_r](r)$ remain in bounded subsets of H (using the bounded convergence theorem in the term involving $e_{x,y}^{3,\varepsilon}$), then

$$(u(t), y) = (U(t,s)\,x, y) + \left(\int_s^t U(t,r)\,g(r)\,dr, y\right), \tag{e}$$

and as this holds for all y in a dense subset of H, the proof is complete.

Remark. The proofs for the restricted case follow the same steps except that there are no error terms and the role of D is played by D_3.

Proof of Lemma 1. If the conclusion does not hold there exists $K < \infty$, $\delta > 0$, $\lambda_n \to \infty$ as $n \to \infty$ and $x \in H$ such that

$$\sup_{\|y\| \leqslant K} |(x, \lambda_n R(\lambda_n) y - y)| > \delta \qquad \text{for all } n;$$

hence $|(x, \lambda_n R(\lambda_n) y_n - y_n)| > \delta$ for all n, for some sequence (y_n), $\|y_n\| \leqslant K$. Since $\{y \in H: \|y\| \leqslant K\}$ is weakly compact, by the Eberlein–Smulian theorem [5] there exists a subsequence (y_{n_k}) of (y_n) such that $(x, y_{n_k}) \to (x, y)$ as $k \to \infty$ for some $y \in H$, $\|y\| \leqslant K$. Hence

$$\begin{aligned} &|(x, \lambda_{n_k} R(\lambda_{n_k}) y_{n_k} - y_{n_k})| \\ &\qquad \leqslant |(x, \lambda_{n_k} R(\lambda_{n_k}) y - y)| + |(x, y - y_{n_k})| + |(x, \lambda_{n_k} R(\lambda_{n_k})(y_{n_k} - y))| \end{aligned}$$

The first and second terms on the right tend to 0 as $k \to \infty$ [9, Lemma 3.2]. For the third term we have

$$\begin{aligned} |(x, \lambda_{n_k} R(\lambda_{n_k})(y_{n_k} - y))| &= |(\lambda_{n_k} R^*(\lambda_{n_k}) x, y_{n_k} - y)| \\ &\leqslant |(\lambda_{n_k} R^*(\lambda_{n_k}) x - x, y_{n_k} - y)| + |(x, y_{n_k} - y)| \\ &\leqslant \|\lambda_{n_k} R^*(\lambda_{n_k}) x - x\| \, 2K + |(x, y_{n_k} - y)|, \end{aligned}$$

where $R^*(\lambda)$ is the resolvent of the adjoint infinitesimal generator [9, Lemma 10.2]. Hence the third term also converges to 0 as $k \to \infty$. Therefore $(x, \lambda_{n_k} R(\lambda_{n_k}) y_{n_k} - y_{n_k}) \to 0$ as $k \to \infty$ and we have a contradiction.

REFERENCES

1. A. AROSIO, Duhamel's principle for temporally inhomogeneous evolution equations in Banach space, *Nonlinear Anal.* **8** (1984), 997–1009.
2. J. M. BALL, Strongly continuous semigroups, weak solutions and the variation of constants formula, *Proc. Amer. Math. Soc.* **63** (1977), 370–373.
3. J. M. BALL AND M. SLEMROD, Feedback stabilization of distributed semilinear control systems, *Appl. Math. Optim.* **5** (1979), 169–179.
4. A. CHOJNOWSKA-MICHALIK, Stochastic differential equations in Hilbert space, *in* "Probability Theory," Vol. 5, pp. 53–74, Banach Center Publications, 1979.
5. N. DUNFORD AND J. T. SCHWARTZ, "Linear Operators," Part I, Interscience, New York, 1966.
6. T. KATO, "Perturbation Theory for Linear Operators," 2nd. ed., Springer–Verlag, Berlin/Heidelberg/New York, 1976.
7. P. KOTELENEZ, A submartingale type inequality with applications to stochastic evolution equations, *Stochastics* **8** (1982), 139–152.
8. M. METIVIER, "Semimartingales," de Gruyter, Berlin/New York, 1982.
9. A. PAZY, "Semigroups of Linear Operators with Applications to Partial Differential Equations," Springer–Verlag, New York/Berlin/Heidelberg/Tokyo, 1983.
10. H. TANABE, "Equations of Evolution," Pitman, London/San Francisco/Melbourne, 1979.
11. J. ZABCZYK, Linear Stochastic systems in Hilbert space: structural properties and limit behavior, Inst. Math., Polish Acad. Sci., preprint No. 236, 1981.